On Noether approach in the cosmological model with scalar and gauge fields: symmetries and the selection rule

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Abstract

In this paper, based on the works of Capozziello et al., we have studied the Noether symmetry approach in the cosmological model with scalar and gauge fields proposed recently by Soda et al. The *correct* Noether symmetries and related Lie algebra are given according to the minisuperspace quantum cosmological model. The Wheeler-De Witt (WDW) equation is presented after quantization and the classical trajectories are then obtained in the semi-classical limit. The oscillating features of the wave function in the cosmic evolution recover the so-called Hartle criterion, and the selection rule in minisuperspace quantum cosmology is strengthened. Then we have realized now the proposition that Noether symmetries select classical universes.

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The Noether symmetry approach [1] in general relativity and cosmology is the endorsement of the crucial role of symmetries in physics [2–6]. As pointed out by Capozziello et al. [2–4], such an approach can provide fundamental consideration on the related models in terms of symmetries, and a selection rule [4] will come into being to recover the classical behaviors of cosmic evolution due to the oscillating features of the cosmological wave function. That is, the correlated region in the configuration space of dynamical variables is selected and a classical observable universe emerges. In the minisuperspace description, the so-called Hartle criterion [7] is applied to the solutions of the Wheeler-De Witt (WDW) equation and selects the classical trajectories [2].

Moreover, the vector field (or gauge field) has been taken into account in cosmology, especially for the early universe [8–10]. In the cosmological models with the vector field(s), the anisotropic hair, perturbation, non-Gaussianity and other aspects are studied extensively. Furthermore, if we are in the case of gauge field, that is, the gauge symmetry is conserved in the system, as proposed by Soda et al. [8–10], the exact anisotropic power-law solutions have been obtained when both the potential function for the scalar field and the gauge kinetic function are exponential type, and an attractor arises for a large parameter region [9]. The anisotropic hair related to the so-called cosmic no-hair theorem and the mechanism of magnetogenesis can also be discussed on this stage.

Furthermore, the Lie point symmetries of second-order partial differential equations, including the Schrödinger and the Klein-Gordon equations have been studied in a general Riemannian space by Paliathanasis et al. [11, 12], and these symmetries are related to the Noether point symmetries of the classical Lagrangian. By employing the Lie symmetries of the WDW equation, a general family of hyperbolic scalar field potentials is also discussed within the framework of perfect fluid in Ref. [11], and the exact solutions of the field equations could be obtained based on the Lie symmetries of the WDW equation.

Then, if we want to discuss the Noether approach in cosmological model with scalar and vector fields, we should start with the following Lagrangian [8, 9]

$$S=\int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R-\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-V(\phi)-\frac{1}{4}f^2(\phi)F_{\mu\nu}F^{\mu\nu}\right], \tag{1}$$

where the flat Robertson-Walker background geometry is assumed, g, R and $a(t)$ are the determinant of the metric, the Ricci scalar and the scale factor respectively, the reduced Planck mass takes $M_p = 1$, while $V(\phi)$ is a potential of the scalar field ϕ and $f(\phi)$ is

the gauge kinetic function, a coupling between the scalar field and the strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for the vector field A_μ. Such a model has been studied recently in Noether symmetry approach by Vakili in Ref. [5]. Here, based on the works of Capozziello et al., we would like to begin anew to provide the *correct* Noether symmetries and related Lie algebra thanks to the minisuperspace quantum cosmological model. Then we try to obtain the solutions of the Wheeler-De Witt (WDW) equation and present the classical trajectories in the semi-classical limit. The Hartle criterion should be applied and the selection rule is reinforced.

As in Refs. [8, 9], we take the vector field A_μ along the x-axis $A_\mu = (0, A_x(t), 0, 0)$, and the scalar field $\phi = \phi(t)$, both of them are considered as homogeneous fields. After some manipulations and do an integration by parts, a point-like Lagrangian is obtained in the following form

$$\mathcal{L} = -3a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 + \frac{1}{2}af^2(\phi)\dot{A}^2 - a^3V(\phi). \tag{2}$$

The configuration space for such a Lagrangian is $\mathcal{Q} \equiv (a, \phi, A)$, and then cosmological dynamics can be reached on such a three dimensional FRW minisuperspace.

According to Capozziello et al. [2], the vector field X

$$X = \alpha^i(q)\frac{\partial}{\partial q^i} + \dot{\alpha}^i(q)\frac{\partial}{\partial \dot{q}^i} \tag{3}$$

could be acted on a Lagrangian $\mathcal{L}$ defined on the tangent space of configurations $T\mathcal{Q}$, then the Lie derivative of $\mathcal{L}$ is

$$L_X\mathcal{L} = X\mathcal{L} = \alpha^i(q)\frac{\partial \mathcal{L}}{\partial q^i} + \dot{\alpha}^i(q)\frac{\partial \mathcal{L}}{\partial \dot{q}^i}. \tag{4}$$

If $L_X\mathcal{L} = 0$, we call X a symmetry for the dynamics of $\mathcal{L}$. It is clear that $\mathcal{L}$ in Eq. (2) does not depend on A, which is then cyclic. So we have the symmetry definitely

$$X_3 = \frac{\partial}{\partial A}, \tag{5}$$

that is, the related α^3 is a constant and independent of variables a, ϕ and A.

Considering the model Eq. (2) presented here, we have

$$X = \alpha^1\frac{\partial}{\partial a} + \alpha^2\frac{\partial}{\partial \phi} + \alpha^3\frac{\partial}{\partial A} + \dot{\alpha}^1\frac{\partial}{\partial \dot{a}} + \dot{\alpha}^2\frac{\partial}{\partial \dot{\phi}} + \dot{\alpha}^3\frac{\partial}{\partial \dot{A}}. \tag{6}$$

If the Lie derivative of $\mathcal{L}$ vanishes along X, that is, $L_X\mathcal{L} = 0$ is satisfied, a constant of motion can be deduced and the Noether theorem holds.

After a straightforward calculation, we get a quadratic expression of a, ϕ and A. Then each coefficient should be zero and a system of partial differential equations for α^i $(i = 1, 2, 3)$ is obtained.

$$\alpha^1 + 2a\frac{\partial \alpha^1}{\partial a} = 0, \tag{7}$$

$$3\alpha^1 + 2a\frac{\partial \alpha^2}{\partial \phi} = 0, \tag{8}$$

$$-6\frac{\partial \alpha^1}{\partial \phi} + a^2\frac{\partial \alpha^2}{\partial a} = 0, \tag{9}$$

$$-6\frac{\partial \alpha^1}{\partial A} + f^2(\phi)\frac{\partial \alpha^3}{\partial a} = 0, \tag{10}$$

$$a^2\frac{\partial \alpha^2}{\partial A} + f^2(\phi)\frac{\partial \alpha^3}{\partial \phi} = 0, \tag{11}$$

$$\alpha^1 f(\phi) + 2a\alpha^2 f'(\phi) + 2af(\phi)\frac{\partial \alpha^3}{\partial A} = 0, \tag{12}$$

$$3\alpha^1 V(\phi) + a\alpha^2 V'(\phi) = 0, \tag{13}$$

where the prime is a derivative with respect to the scalar field ϕ.

To recover the symmetry Eq. (5), it requires that α^3 is independent of variables a, ϕ and A. Then from Eq. (10), $\partial\alpha^1/\partial A = 0$, and due to Eq. (7),

$$\alpha^1(a, \phi) = a^{-1/2}F(\phi), \tag{14}$$

where F is an arbitrary function dependent of ϕ only. From Eq. (11), $\partial\alpha^2/\partial A = 0$, and due to Eq. (8),

$$\alpha^2(a, \phi) = -\frac{3}{2}a^{-3/2}\int F(\phi)d\phi. \tag{15}$$

From Eq. (12), $\alpha^1 f + 2a\alpha^2 f' = 0$, and

$$\frac{f'}{f} = -\frac{\alpha^1}{2a\alpha^2}. \tag{16}$$

From Eq. (13), $3\alpha^1 V + a\alpha^2 V' = 0$, and

$$\frac{V'}{V} = -\frac{3\alpha^1}{a\alpha^2}. \tag{17}$$

Then from Eq. (9),

$$F' - \frac{3}{8}\int F d\phi = 0, \tag{18}$$

we can obtain the following solution for $F(\phi)$

$$F(\phi) = b_1 e^{\omega\phi} + b_2 e^{-\omega\phi}, \tag{19}$$

where b_1 and b_2 are constants of integration, and $\omega^2 = 3/8$. The expressions for α^1 and α^2 are also found

$$\alpha^1 = a^{-1/2}\left(b_1 e^{\omega\phi} + b_2 e^{-\omega\phi}\right), \tag{20}$$

$$\alpha^2 = -\frac{3}{2\omega} a^{-3/2}\left(b_1 e^{\omega\phi} - b_2 e^{-\omega\phi}\right). \tag{21}$$

It should be emphasized that there is no A in the expressions of α^1 and α^2, and the symmetry Eq. (5) is recovered. From Eq. (16)

$$\frac{f'}{f} = \frac{1}{8\omega}\frac{b_1 e^{\omega\phi} + b_2 e^{-\omega\phi}}{b_1 e^{\omega\phi} - b_2 e^{-\omega\phi}}, \tag{22}$$

then the solution of $f(\phi)$ is

$$f(\phi) = \left(b_1 e^{\omega\phi} - b_2 e^{-\omega\phi}\right)^{1/3}. \tag{23}$$

From Eq. (17)

$$\frac{V'}{V} = \frac{3}{4\omega}\frac{b_1 e^{\omega\phi} + b_2 e^{-\omega\phi}}{b_1 e^{\omega\phi} - b_2 e^{-\omega\phi}}, \tag{24}$$

then the solution of $V(\phi)$ is

$$V(\phi) = \left(b_1 e^{\omega\phi} - b_2 e^{-\omega\phi}\right)^2. \tag{25}$$

So we reach a basis of symmetries on $T\mathcal{Q}$,

$$X_1 = a^{-1/2}e^{-\omega\phi}\frac{\partial}{\partial a} + \frac{3}{2\omega}a^{-3/2}e^{-\omega\phi}\frac{\partial}{\partial\phi} + \left(a^{-1/2}e^{-\omega\phi}\right)^{\cdot}\frac{\partial}{\partial\dot{a}} + \left(\frac{3}{2\omega}a^{-3/2}e^{-\omega\phi}\right)^{\cdot}\frac{\partial}{\partial\dot{\phi}}, \tag{26}$$

$$X_2 = a^{-1/2}e^{\omega\phi}\frac{\partial}{\partial a} - \frac{3}{2\omega}a^{-3/2}e^{\omega\phi}\frac{\partial}{\partial\phi} + \left(a^{-1/2}e^{\omega\phi}\right)^{\cdot}\frac{\partial}{\partial\dot{a}} - \left(\frac{3}{2\omega}a^{-3/2}e^{\omega\phi}\right)^{\cdot}\frac{\partial}{\partial\dot{\phi}}, \tag{27}$$

$$X_3 = \frac{\partial}{\partial A}. \tag{28}$$

The corresponding constants of motion associated with the above symmetries are

$$K_1 = \frac{3}{2}a^{1/2}e^{-\omega\phi}\dot{a} - \omega a^{-3/2}e^{-\omega\phi}\dot{\phi}, \tag{29}$$

$$K_2 = \frac{3}{2}a^{1/2}e^{\omega\phi}\dot{a} + \omega a^{3/2}e^{\omega\phi}\dot{\phi}, \tag{30}$$

$$K_3 = af^2\dot{A}. \tag{31}$$

It can be shown that all the symmetries are commuted with each other, that is, the following Lie algebra is satisfied,

$$[X_i, X_j] = 0, \qquad i,j = 1,2,3. \tag{32}$$

And the corresponding constants of motion also close the same algebra in terms of Poisson bracket,

$$[K_i, K_j] = 0, \qquad i, j = 1, 2, 3. \tag{33}$$

According to Capozziello et al. [2], to identify the cyclic variables, we make a transformation on the vector field Eq. (6) (or Eq. (26)) for X_1,

$$w = a^{3/2} e^{\omega\phi}, \qquad z = a^{3/2} e^{-\omega\phi}, \qquad A = A, \tag{34}$$

to satisfied that

$$i_{X_1} dw = 3, \qquad i_{X_1} dz = 0, \qquad i_{X_1} dA = 0. \tag{35}$$

The configuration space is transformed from $\mathcal{Q} \equiv (a, \phi, A)$ to $\tilde{\mathcal{Q}} \equiv (w, z, A)$ correspondingly.

Then the Lagrangian Eq. (2) transformed into

$$\mathcal{L} = -\frac{4}{3}\dot{w}\dot{z} - z^2 + \frac{1}{2} z^{2/3} \dot{A}^2, \tag{36}$$

and the conjugated momenta of dynamic variables w, z and A are respectively

$$\pi_w = \frac{\partial \mathcal{L}}{\partial \dot{w}} = -\frac{4}{3}\dot{z}, \tag{37}$$

$$\pi_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = -\frac{4}{3}\dot{w}, \tag{38}$$

$$\pi_A = \frac{\partial \mathcal{L}}{\partial \dot{A}} = z^{2/3}\dot{A}. \tag{39}$$

In the Hamiltonian dynamics, we should get the Hamiltonian firstly

$$\mathcal{H} = -\frac{3}{4}\pi_w \pi_z + \frac{1}{2} z^{-2/3} \pi_A^2 + z^2, \tag{40}$$

and due to the Noether symmetries, the constants of motion are

$$\pi_w = \Sigma_0, \qquad \pi_A = \Sigma_1. \tag{41}$$

We can quantize the system as follows,

$$\pi_w \to \hat{\pi}_w = -i\partial_w, \qquad \pi_z \to \hat{\pi}_z = -i\partial_z, \qquad \pi_A \to \hat{\pi}_A = -i\partial_A, \tag{42}$$

$$\mathcal{H} \to \hat{\mathcal{H}}(q^j, -i\partial_{q^j}), \tag{43}$$

then the Wheeler-De Witt (WDW) equation could be obtained

$$\left[-\frac{3}{4}(-i\partial_w)(-i\partial_z) + \frac{1}{2} z^{-2/3}(-i\partial_A)^2 + z^2\right] \mid \Psi\rangle = 0. \tag{44}$$

Separating the variables via the constants of motion,

$$| \Psi\rangle =| \Omega(w)\rangle \mid \chi(A)\rangle \mid \xi(z)\rangle \propto e^{i\Sigma_0 w} e^{i\Sigma_1 A} \mid \xi(z)\rangle, \tag{45}$$

the solution of $\xi(z)$ is found

$$\xi(z) = \exp\left[i\frac{2\Sigma_1^2}{\Sigma_0} z^{1/3} + i\frac{4}{9\Sigma_0} z^3\right]. \tag{46}$$

Such a wave function with the oscillating feature recovers the so-called Hartle criterion [7] and allows us to get a classical observable universe.

If we now take the action (or the Hamiltonian principal function) as

$$S = \Sigma_0\omega + \Sigma_1 A + \frac{2\Sigma_1^2}{\Sigma_0} z^{1/3} + \frac{4}{9\Sigma_0} z^3, \tag{47}$$

the corresponding momenta which conjugate the generalized coordinates are obtained

$$\pi_w = \frac{\partial S}{\partial w} = \Sigma_0, \tag{48}$$

$$\pi_z = \frac{\partial S}{\partial z} = \frac{2\Sigma_1^2}{3\Sigma_0} z^{-2/3} + \frac{4}{3\Sigma_0} z^2, \tag{49}$$

$$\pi_A = \frac{\partial S}{\partial A} = \Sigma_1. \tag{50}$$

Separating the variables via above formulae, we can solve Hamilton-Jacobi (HJ) equation in the semi-classical limit. The classical trajectories in the configuration space, $\tilde{\mathcal{Q}} \equiv (w, z, A)$, then can be expressed

$$z(t) = k_1 t + k_2, \tag{51}$$

$$A(t) = c(k_1 t + k_2)^{1/3} + A_0, \tag{52}$$

$$w(t) = c_1 + c_2(k_1 t + k_2)^{1/3} + c_3(k_1 t + k_2)^3, \tag{53}$$

where k_2, A_0 and c_1 are constants of integration, while k_1, c, c_2 and c_3 are dependent of constants of motion Σ_0 and Σ_1. If we go back to $\mathcal{Q} \equiv (a, \phi, A)$, we can get the classical cosmological solutions,

$$a(t) = \left[d_1(t-t_0) + d_2(t-t_0)^{4/3} + d_3(t-t_0)^4\right]^{1/3}, \tag{54}$$

$$\phi(t) = -\frac{1}{\omega} \ln \frac{k_1 t + k_2}{\left[d_1(t-t_0) + d_2(t-t_0)^{4/3} + d_3(t-t_0)^4\right]^{1/2}}, \tag{55}$$

where t_0 and d_i $(i = 1, 2, 3)$ could be looked on as constants of integration and should be determined by observations. Then the oscillating regime is selected and the classical

behaviors are recovered. One of the main results is that the late time acceleration could be obtained from the classical solution, and the roll of scalar field may be more important than that of vector field as discussed in Ref. [5]. As to the Theorem in Ref. [4], Noether symmetries select classical universes. Such a picture has now been realized here in the cosmological model with vector field(s) proposed by Soda et al [8, 9].

Moreover, the approach given here can be generalized naturally and consistently to the case of phantom field [13] and other cases (such as the so-called quintom field) rather than other approaches. If we take the following Lagrangian,

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R - \epsilon\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi) - \frac{1}{4}f^2(\phi)F_{\mu\nu}F^{\mu\nu}\right], \tag{56}$$

then $\epsilon = +1$ is the standard case just discussed here, while $\epsilon = -1$ is the phantom field case [14]. Same approach could be applied naturally though different behaviors are expected, and related energy conditions are also to be discussed. Furthermore, the related WDW equation is more complex, then we shall discuss the Lie symmetries of the WDW equation as those in Ref. [11] to identify the properties of the solution. All these should be prepared systematically together with various other aspects and will be placed somewhere else.

In summary, we have realized the proposition that Noether symmetries select classical universes in the cosmological model with vector (gauge) field(s). Based on the correct Noether symmetries, we give the related Lie algebras for symmetries and constants of motion according to the minisuperspace quantum cosmological model. We solve the Wheeler-De Witt (WDW) equation after quantization and select a subset of the solutions with oscillating behaviors. Then the classical observable universe is recovered.

Finally, I would like to give a brief comment on arXiv:1410.3131 [5].

Though it is an interesting work, frankly, the symmetries in arXiv:1410.3131 are incorrect.

Firstly and evidently, $\frac{\partial}{\partial A}$ is *not* a symmetry there, but in fact it is. While the constant of motion related to this symmetry, called P_{0A} or P_A, is used in the calculation and plays a crucial role to obtain the final results and conclusions. It is contradictive. We know that a constant of motion should be corresponded to a symmetry. If there is no such a symmetry, we cannot get the related constant of motion logically.

Secondly, other symmetries are also not correct, and no related constants of motion can be given. The follow-up calculations seem not to be directly based on these constants of motion according to the method used in the paper, the incorrectness looks not so evident

like the one above. So the Lie Algebra is not given, in fact cannot be given there, nor the WDW equation and so on.

Importantly, if the constant of motion used in the paper, called P_{0A} or P_A, could be obtained, it should be satisfied that, based on its own derivation,

$$\dot{\gamma} = \frac{1}{2}\dot{a}\gamma_0 a^{-\frac{1}{2}} e^{\mp\omega\phi/3} \mp \frac{\omega\dot{\phi}}{3}\gamma_0 a^{\frac{1}{2}} e^{\mp\omega\phi/3} = 0, \tag{57}$$

that is,

$$-\frac{3}{2}\frac{\dot{a}}{a} \pm \omega\dot{\phi} = 0. \tag{58}$$

While the one in the paper, eq. (38), is (the quantity Q is set to be zero)

$$-\frac{3}{2}\frac{\dot{a}}{a} \pm \omega\dot{\phi} + 3\frac{\dot{A}}{A} = 0. \tag{59}$$

It is clearly that $\frac{\dot{A}}{A} \neq 0$, so no any result like P_{0A} or P_A given in the paper can be obtained.

The above equation is crucial to obtain the final results. Without such an equation, no any of the final results shown in the paper could be obtained. Taking some specific values for the constants, the equations of the system become solvable and the results look like correct ones. The results just happen to be superficially, the calculations are incorrect.

The correct symmetries and the related Lie algebra are given in the present paper, the cosmological variables are calculated by a different method based on the works of Capozziello et al.

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